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European Journal of Mechanics B/Fluids 24 (2005) 468–477



Drag force acting on a bubble in a cloud of compressible spherical bubbles at large Reynolds numbers

S.L. Gavriluk^{a,b,*}, V.M. Teshukov^c

^a *Laboratoire de Modélisation en Mécanique et Thermodynamique, Faculté des Sciences et Techniques de Saint-Jérôme, Case 322, Avenue Escadrille Normandie-Niemen, 13397 Marseille cedex 20, France*

^b *CNRS UMR 6595, IUSTI, 5, rue E. Fermi, 13453 Marseille cedex 13, France*

^c *Lavrentyev Institute of Hydrodynamics, Lavrentyev prospect 15, Novosibirsk 630090, Russia*

Received 2 September 2004; received in revised form 13 December 2004; accepted 16 December 2004

Available online 29 January 2005

Abstract

We have derived the expressions for the viscous forces acting on a bubble in a cloud of bubbles by using the approach by Levich. To obtain the dissipation function, an approximate expression for the velocity potential calculated previously by the authors up to order β^3 has been used. Here $\beta = \bar{b}/d$ is a small dimensionless parameter, \bar{b} is the mean bubble radius and d is the mean distance between bubbles.

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Keywords: Bubbly fluids; Lagrange's formalism; Levich's approach

1. Introduction

Levich [1] calculated the drag force acting on a bubble moving through a liquid at large Reynolds and small Weber numbers (the last condition guarantees that the bubble shape is spherical) defining the total viscous dissipation through the velocity potential of irrotational flow (see also Moore [2] and Batchelor [3]). With the use of this approach, he obtained the drag force F in a translational motion of a bubble of unchanged radius

$$F = 12\pi\mu bU. \quad (1)$$

It differs from the drag force given by the theory of viscous potential motions. In the last method, the viscous flow is supposed to be potential, and the viscosity is taken into account only in the dynamic condition expressing the continuity of the normal stress at the gas–liquid interface (Moore [4], see also Joseph and Wang [5]). The corresponding drag force is then

$$F = 8\pi\mu bU.$$

Here μ is the dynamic viscosity of the fluid, U is the bubble velocity, and b is the bubble radius. Moore [2] justified further the approach by Levich by developing of the model of the boundary layer wrapped around bubble. Moreover, recently Magnaudet and Legendre [6] calculated the drag force on a spherical compressible bubble by using the full Navier–Stokes equations. In

* Corresponding author.

E-mail addresses: sergey.gavriluk@univ.u-3mrs.fr (S.L. Gavriluk), teshukov@hydro.nsc.ru (V.M. Teshukov).

particular, they found that the Levich formula (1) is valid not only in the high-Reynolds limit, but also for moderate and even small Reynolds numbers, provided the bubble pulsation velocity is high with respect to the flow velocity.

Both approaches, the method of dissipation function and the method of viscous potential motions, give the same result if we calculate the viscous forces on an oscillating bubble without translational motion (the Rayleigh–Plesset equation):

$$b \frac{d^2 b}{dt^2} + \frac{3}{2} \left(\frac{db}{dt} \right)^2 = \frac{1}{\rho} (p_g - p_\infty) - \frac{4\mu}{\rho b} \frac{db}{dt}. \quad (2)$$

Here ρ is the fluid density, $p_g(b)$ is the pressure in a bubble (it is a given function of b), and p_∞ is the fluid pressure at infinity.

The Levich approach allows to apply the Lagrange formalism for constructing the governing equations of motion with non-potential forces derived from the Rayleigh dissipation function method (see, for example, Goldstein, Poole and Safko [7]). More exactly, if we consider a bubble of radius $b(t)$ oscillating with the velocity $s(t) = db/dt$ whose center position is $\mathbf{r}(t)$ and the translational velocity $\mathbf{v}(t) = d\mathbf{r}(t)/dt$, the equations of motion are (Voinov and Golovin [8]):

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} \right) - \frac{\partial L}{\partial \mathbf{r}} &= -\frac{1}{2} \frac{\partial \Phi}{\partial \mathbf{v}}, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial s} \right) - \frac{\partial L}{\partial b} &= -\frac{1}{2} \frac{\partial \Phi}{\partial s}. \end{aligned} \quad (3)$$

Eqs. (3) imply the energy equation in the form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} \mathbf{v} + \frac{\partial L}{\partial s} s - L \right) = -\Phi.$$

Here L is the Lagrangian of the system, Φ is the Rayleigh dissipation function. In the case of a massless bubble, the Lagrangian is

$$L = \frac{\pi \rho b^3}{3} |\mathbf{v}|^2 + 2\pi \rho b^3 s^2 - \varepsilon(\tau).$$

Here

$$\varepsilon(\tau) = \int_{\tau}^{\infty} p_g(z) dz + p_\infty \tau$$

is the internal energy of the gas–liquid system, $\tau = 4\pi b^3/3$ is the bubble volume. The dissipation function Φ can be explicitly calculated for the velocity potential of the flow around a single bubble:

$$\varphi = -\frac{b^2 s}{|\mathbf{x} - \mathbf{r}(t)|} + \frac{b^3}{2} \mathbf{v} \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{r}(t)|} \right).$$

It is given by

$$\Phi = 2\mu \int_{R^3 \setminus B} D : D d\omega = -\mu \int_{\Gamma} \frac{\partial}{\partial n} |\nabla \varphi|^2 d\Gamma = 4\pi \mu b (4s^2 + 3|\mathbf{v}|^2). \quad (4)$$

Here B is a region occupied by the bubble, $\Gamma = \partial B$ is the bubble boundary, $D = (\partial \mathbf{u}/\partial \mathbf{x} + (\partial \mathbf{u}/\partial \mathbf{x})^T)/2$ is the rate of deformation tensor, $\mathbf{u} = \nabla \varphi$, \mathbf{n} is the normal vector to Γ directed to the fluid. System (3) can be rewritten in the following equivalent form:

$$\begin{aligned} \frac{d}{dt} \left(\frac{2}{3} \pi \rho b^3 \mathbf{v} \right) &= -12\pi \mu b \mathbf{v}, \\ b \frac{d^2 b}{dt^2} + \frac{3}{2} \left(\frac{db}{dt} \right)^2 - \frac{1}{4} |\mathbf{v}|^2 &= \frac{1}{\rho} (p_g(b) - p_\infty) - \frac{4\mu}{\rho b} \frac{db}{dt}. \end{aligned} \quad (5)$$

The general case of N interacting rigid bubbles was considered first by Golovin [9]. For a dilute bubbly mixture, he derived the equations of motion as well as the friction forces (by using Levich's approach) up to order β^3 . Here $\beta = \bar{b}/d$ is a small parameter which is the ratio of the mean bubble radius \bar{b} to the mean distance d between bubbles. Obviously, $4\pi\beta^3/3$ is the volume fraction of bubbles in the limit $N \rightarrow \infty$. Kok [10,11] (and references therein), studied analytically and experimentally the dynamics of a pair of rigid bubbles of equal radii. In particular, he derived the equations of motion up to any order of β . Sangani and Didwania [12] and Smereka [13] examined the dynamics of N bubbles numerically. In particular, they showed that

the random state of massless rigid bubbles was unstable, and that the bubbles formed aggregates in plane transverse to gravity. This conclusion was confirmed analytically by Van Wijngaarden [14]. The “thermodynamics” of clustering of a system of rigid bubbles has been studied by Yurkovetsky and Brady [15]. They proved that sufficiently strong bubble velocity oscillations can prevent clustering. Russo and Smereka [16] and Herrero, Lucquin-Desreux and Perthame [17] obtained a Vlasov type system describing the motion of massless rigid spheres. Harkin, Kaper and Nadim [18] obtained the equations of motion of two pulsating and translating spherical bubbles. They calculated the flow potential up to order β^4 . The friction forces were not calculated with the same precision: the authors used for numerical purposes the classical Levich forces acting on an isolated bubble. Teshukov and Gavriluk [19] studied the system of N compressible bubbles and obtained a Vlasov type system in the dissipation-free limit. In that work, they calculated the flow potential up to order β^3 , which accounts for pair-wise interactions between bubbles. They have shown, in particular, that the bubble oscillations regularize the bubble flow dynamics: in the hydrodynamic limit, the dispersion relation has real roots. One can speculate that this could be responsible for the stability of random bubble clouds. In that work, the expressions for the friction forces were not derived.

The aim of this article is to determine the drag forces on bubbles in a cloud of N massless compressible bubbles. This problem was considered by Voinov and Golovin [8] and Wang and Smereka [20] who also obtained approximate expressions of these forces. In the present work, the expressions for the drag forces are obtained using Levich’s approach and an explicit formula for the velocity potential of the flow with N compressible bubbles calculated up to order β^3 in Teshukov and Gavriluk [19].

2. Velocity potential for a system of N compressible bubbles in an incompressible fluid

In the Levich approach, the dissipation function Φ is calculated by using the velocity potential for a system of N bubbles. The corresponding potential has been found in Teshukov [21] and Teshukov and Gavriluk [19] in explicit asymptotic form which is correct up to order β^3 . For completeness, we shall shortly mention here results obtained in the last article.

We consider N spherical gas bubbles moving in an unbounded inviscid incompressible fluid. We assume that the fluid flow in the region between bubbles is irrotational and the velocity vector vanishes as $|\mathbf{x}| \rightarrow \infty$. The fluid velocity potential $\varphi(t, \mathbf{x})$ is a solution of the following boundary-value problem:

$$\Delta\varphi = 0, \quad \mathbf{x} \in \Omega = R^3 \setminus \bigcup_j B_j, \quad \frac{\partial\varphi}{\partial n} \Big|_{\Gamma_j} = \mathbf{v}_j \cdot \mathbf{n}_j + s_j, \quad (6)$$

$$\nabla\varphi \rightarrow 0, \quad |\mathbf{x}| \rightarrow \infty.$$

Here $\mathbf{v}_j(t) = \mathbf{x}'_j(t)$ are the velocities of the centers of spherical bubbles, $s_j = b'_j(t)$ are the velocities of expansion of the bubbles, $\mathbf{x}_j(t)$ are the radius-vectors of the centers, $b_j(t)$ are the bubble radii ($j = 1, \dots, N$), B_j and Γ_j are a ball and a sphere of radius $b_j(t)$ with center at the point $\mathbf{x}_j(t)$, respectively, and \mathbf{n}_j is the unit normal vector to Γ_j directed to the fluid. Here the prime denotes the derivative with respect to time. This problem describes the motion of a collection of compressible bubbles in otherwise quiescent fluid.

The unknown potential of the irrotational flow can be written as

$$\varphi = \sum_{j=1}^N (\mathbf{v}_j \cdot \boldsymbol{\psi}_j + s_j \varphi_j),$$

where, in view of (6), the harmonic functions $\boldsymbol{\psi}_j(t, \mathbf{x})$ and $\varphi_j(t, \mathbf{x})$ satisfy the conditions

$$\frac{\partial\boldsymbol{\psi}_j}{\partial n} \Big|_{\Gamma_k} = \delta_{jk} \mathbf{n}_j, \quad \frac{\partial\varphi_j}{\partial n} \Big|_{\Gamma_k} = \delta_{jk}$$

and their gradients vanish at infinity. Here δ_{jk} are the Kronecker symbols.

In what follows, we consider a rarefied bubbly fluid for which β is a small parameter. Asymptotic expressions of φ_j and $\boldsymbol{\psi}_j$ up to order β^3 are given in Teshukov and Gavriluk [19]:

$$\varphi_j = -\frac{b_j^2}{r_j} + \sum_{i \neq j} \varphi_{ji}, \quad (7)$$

$$\varphi_{ji} = -\frac{b_j^2}{r_{ij}} \sum_{n=1}^2 \frac{n}{n+1} \left(\frac{b_i}{r_i}\right)^{n+1} \left(\frac{b_i}{r_{ij}}\right)^n P_n(\cos\theta_{ji}), \quad (8)$$

$$\psi_j = \frac{b_j^3}{2} \nabla(r_j^{-1}) + \sum_{i \neq j} \psi_{ji}, \quad (9)$$

$$\psi_{ji} = -\frac{b_j}{4} \left(\frac{b_j}{r_{ij}} \right)^2 \left(\frac{b_i}{r_{ij}} \right) \left(\frac{b_i}{r_i} \right)^2 B_{ij} \frac{\mathbf{x} - \mathbf{x}_i}{r_i}. \quad (10)$$

Here

$$r_i = |\mathbf{x} - \mathbf{x}_i|, \quad r_{ji} = |\mathbf{x}_j - \mathbf{x}_i|, \quad \mathbf{n}_{ji} = \frac{\mathbf{x}_j - \mathbf{x}_i}{r_{ji}}, \quad \cos \theta_{ji} = \frac{(\mathbf{x} - \mathbf{x}_i) \cdot \mathbf{n}_{ji}}{r_i}, \quad B_{ij} = I - 3\mathbf{n}_{ji} \otimes \mathbf{n}_{ji},$$

$P_n(\cos \theta)$ are the Legendre polynomials (Gradsteyn and Ryzhik [22]), I is the unit matrix, and $\mathbf{a} \otimes \mathbf{b}$ is the tensor product.

It is easy to see that $\partial \varphi_j / \partial n|_{\Gamma_k} = \delta_{jk} + O(\beta^4)$ by virtue of the estimates $\varphi_{ji}|_{\Gamma_k} = O(\beta^4)$ and $\nabla \varphi_{ji}|_{\Gamma_k} = O(\beta^4)$, $i \neq k$. The harmonic function ψ_{ji} satisfies the conditions $\partial(\psi_{j0} + \psi_{ji}) / \partial n|_{\Gamma_i} = 0$, $\psi_{ji}|_{\Gamma_k} = O(\beta^4)$, $\partial \psi_j / \partial n|_{\Gamma_k} = \delta_{jk} \mathbf{n}_j + O(\beta^4)$, and $\nabla \psi_{ji}|_{\Gamma_k} = O(\beta^4)$ for $i \neq k$.

In Teshukov and Gavriluk [19] there is a misprint: the sign “–” before the expression (10) was absent. However, all further calculations in the above mentioned paper took into account the right sign.

Using the asymptotic expansion in β , we have obtained an approximate representation of the fluid velocity potential for the specified positions \mathbf{x}_j , radii b_j , translational velocities \mathbf{v}_j and dilatation velocities s_j of bubbles. This enables us to calculate the kinetic energy of the fluid:

$$T = \frac{\rho}{2} \left(\sum_{i=1}^N b_i^3 \left(\frac{2}{3} \pi |\mathbf{v}_i|^2 + 4\pi s_i^2 \right) + \sum_{i=1}^N \sum_{j \neq i} \left(4\pi b_i^2 b_j \frac{b_j}{r_{ij}} s_i s_j + 2\pi b_i^3 \left(\frac{b_j}{r_{ij}} \right)^2 s_j (\mathbf{n}_{ji} \cdot \mathbf{v}_i) + 2\pi b_j^3 \left(\frac{b_i}{r_{ij}} \right)^2 s_i (\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \pi b_i^3 \left(\frac{b_j}{r_{ij}} \right)^3 (\mathbf{v}_j \cdot B_{ij} \mathbf{v}_i) \right) \right). \quad (11)$$

In this formula, the first sum comprises the zero order terms in β and the second sum includes terms of orders β , β^2 , and β^3 . The lower order approximate formula for T containing the terms up to β^2 has also been obtained by Voinov and Golovin [8]. For the case of rigid bubbles of equal radii the formula for T was obtained by Golovin [9] (see also Russo and Smereka [16]).

The problem of motion of a fluid with bubbles is Lagrangian in the case where the flow of the fluid is completely determined by the bubble motion (Voinov and Golovin [8]). The Euler–Lagrange equations for the generalized coordinates and corresponding velocities are:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{y}}_j} \right) - \frac{\partial L}{\partial \mathbf{y}_j} = 0, \quad L = T - U, \quad U = \sum_{j=1}^N \varepsilon(\tau_j). \quad (12)$$

Here $\mathbf{y}_j = (\mathbf{x}_j, b_j)^T$ ($j = 1, \dots, N$) are the vectors with four components. Now we will introduce the drag forces by using the Levich approach.

3. Dissipation function

The drag forces are calculated through the derivatives $\partial \Phi / \partial \mathbf{v}_j$, $\partial \Phi / \partial s_j$ of the dissipation function

$$\Phi = 2\mu \int_{R^3 \setminus \bigcup_j B_j} D : D \, d\omega = 2\mu \int_{R^3 \setminus \bigcup_j B_j} \frac{\partial^2 \varphi}{\partial \mathbf{x}^2} : \frac{\partial^2 \varphi}{\partial \mathbf{x}^2} \, d\omega$$

which is a quadratic function of \mathbf{v}_i and s_i . This formula can be rewritten in the form (Batchelor [3]):

$$\Phi = -\mu \sum_{k=1}^N \int_{\Gamma_k} \frac{\partial}{\partial n} |\nabla \varphi|^2 \, d\Gamma = -\mu \sum_{k=1}^N \int_{\Gamma_k} \frac{\partial}{\partial r} |\nabla \varphi|^2 \, d\Gamma.$$

Here r is the distance from the bubble center. We assume that the system of N bubbles is governed by the equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{y}}_j} \right) - \frac{\partial L}{\partial \mathbf{y}_j} = -\frac{1}{2} \frac{\partial \Phi}{\partial \dot{\mathbf{y}}_j}, \quad \mathbf{y}_j = (\mathbf{x}_j, b_j)^T \quad (13)$$

generalizing (12) to the case of non-potential forces. To find the drag forces explicitly, we need to calculate the derivatives of Φ with respect to \mathbf{v}_j and s_j . For any constant vector \mathbf{h}

$$\begin{aligned}\frac{\partial \Phi}{\partial \mathbf{v}_j} \cdot \mathbf{h} &= -2\mu \sum_{k=1}^N \int_{\Gamma_k} \frac{\partial}{\partial n} (\nabla(\boldsymbol{\psi}_j \cdot \mathbf{h}) \cdot \nabla \varphi) d\Gamma = -4\mu \sum_{k=1}^N \int_{\Gamma_k} \nabla(\boldsymbol{\psi}_j \cdot \mathbf{h}) \cdot \frac{\partial}{\partial r} (\nabla \varphi) d\Gamma \\ &= -4\mu \sum_{k=1}^N \int_{\Gamma_k} \mathbf{h}^T \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \frac{\partial}{\partial r} (\nabla \varphi) d\Gamma.\end{aligned}$$

Hence

$$\left(\frac{\partial \Phi}{\partial \mathbf{v}_j} \right)^T = -4\mu \sum_{k=1}^N \int_{\Gamma_k} \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \frac{\partial}{\partial r} (\nabla \varphi) d\Gamma. \quad (14)$$

In the same manner,

$$\frac{\partial \Phi}{\partial s_j} = -4\mu \sum_{k=1}^N \int_{\Gamma_k} \nabla \varphi_j \cdot \frac{\partial}{\partial r} (\nabla \varphi) d\Gamma. \quad (15)$$

To calculate (14), (15), we evaluate the values of $\boldsymbol{\psi}_j$ and φ_j as well as the gradients of these functions on all Γ_i . In the following the outward normal vector \mathbf{n}_j to Γ_j , $j = 1, \dots, N$, will be denoted as \mathbf{n} . From (7)–(10) we obtain approximate expressions of these functions and their derivatives correct up to β^3 which are necessary to construct the dissipation function. Further, we will use the sign “ \approx ” to say that a given expression is correct up to β^3 (terms of order β^4 and higher are omitted).

3.1. Calculation of drag forces

In this section we calculate the drag forces and give an expression of the dissipation function in explicit form.

Using the approximate formulae for $\boldsymbol{\psi}_j$ and φ_j obtained in Appendices A and B, we can evaluate the derivatives of the dissipation function with respect to \mathbf{v}_j and s_j differing only by a constant multiplier from the drag forces. First, we shall calculate the drag forces in the equations governing the translational motion of bubbles. The terms up to β^3 in (14) are given by

$$\begin{aligned}\left(\frac{\partial \Phi}{\partial \mathbf{v}_j} \right)^T &\approx -4\mu \int_{\Gamma_j} \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \sum_{i=1}^N \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma \\ &\quad - 4\mu \sum_{i \neq j} \int_{\Gamma_i} \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma.\end{aligned} \quad (16)$$

To evaluate (16) we consider three integrals. The first one is easily evaluated after substituting of approximate expressions for the derivatives of φ_j and $\boldsymbol{\psi}_j$ (see Appendices A and B):

$$\begin{aligned}V_1 &= -4\mu \int_{\Gamma_j} \frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \left(s_j \frac{\partial \nabla \varphi_j}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_j \right) d\Gamma = 4\mu \int_{\Gamma_j} \frac{1}{2} (I - 3\mathbf{n} \otimes \mathbf{n}) \left(s_j \mathbf{n} + \frac{3}{2b_j} (I - 3\mathbf{n} \otimes \mathbf{n}) \mathbf{v}_j \right) d\Gamma \\ &= \frac{6\mu}{b_j} \int_{\Gamma_j} (I + 3\mathbf{n} \otimes \mathbf{n}) \mathbf{v}_j d\Gamma = 24\mu\pi b_j \mathbf{v}_j.\end{aligned} \quad (17)$$

We used here the equalities

$$\int_{\Gamma_j} \mathbf{n} d\Gamma = 0, \quad \frac{1}{4\pi b_j^2} \int_{\Gamma_j} \mathbf{n} \otimes \mathbf{n} d\Gamma = \frac{I}{3}. \quad (18)$$

Further, we substitute approximate expressions for φ_j and $\boldsymbol{\psi}_j$ into

$$\begin{aligned}
V_2 &= -4\mu \int_{\Gamma_j} \frac{\partial \psi_j}{\partial \mathbf{x}} \sum_{i \neq j} \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \psi_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma \\
&= 2\mu \int_{\Gamma_j} (I - 3\mathbf{n} \otimes \mathbf{n}) \sum_{i \neq j} \left(\frac{3}{4b_j} \left(\frac{b_i}{r_{ij}} \right)^3 (I - 3\mathbf{n} \otimes \mathbf{n}) B_{ij} \mathbf{v}_i + s_i \left(-\frac{b_i^2}{b_j^2 r_{ij}} \sum_{n=1}^2 n(2n+1) \left(\frac{b_j}{r_{ij}} \right)^n P_n(\cos \theta_{ij}) \mathbf{n} \right. \right. \\
&\quad \left. \left. + \frac{b_i^2}{b_j^2 r_{ij}} \sum_{n=1}^2 \frac{2n+1}{n+1} \left(\frac{b_j}{r_{ij}} \right)^n P'_n(\cos \theta_{ij}) (I - \mathbf{n} \otimes \mathbf{n}) \mathbf{n}_{ij} \right) \right) d\Gamma.
\end{aligned} \tag{19}$$

Taking into account the equalities

$$\int_{\Gamma_j} (I - 3\mathbf{n} \otimes \mathbf{n})(I - 3\mathbf{n} \otimes \mathbf{n}) d\Gamma = 8\pi b_j^2 I, \quad \int_{\Gamma_j} (I - 3\mathbf{n} \otimes \mathbf{n})(I - \mathbf{n} \otimes \mathbf{n}) d\Gamma = \frac{8\pi}{3} b_j^2 I$$

and the definition of the first Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2},$$

we obtain the expression

$$V_2 = 12\pi \mu b_j \sum_{i \neq j} \left(\frac{b_i}{r_{ij}} \right)^3 B_{ij} \mathbf{v}_i + 8\pi \mu b_j \sum_{i \neq j} s_i \left(\frac{b_i}{r_{ij}} \right)^2 \mathbf{n}_{ij} + 12\mu \sum_{i \neq j} \frac{s_i b_i^2}{b_j r_{ij}^2} \int_{\Gamma_j} \mathbf{n} \cos \theta_{ij} d\Gamma.$$

To calculate the last integral, it is convenient to use the spherical coordinates $r_i, \theta_{ij}, \lambda_{ij}$ with z -axis directed along \mathbf{n}_{ij} . In the spherical coordinates the normal vector to Γ_j is:

$$\mathbf{n} = (\sin \theta_{ij} \cos \lambda_{ij}, \sin \theta_{ij} \sin \lambda_{ij}, \cos \theta_{ij}).$$

Using this formula we find

$$\int_{\Gamma_j} \mathbf{n} \cos \theta_{ij} d\Gamma = \frac{4}{3} \pi b_j^2 \mathbf{n}_{ij}$$

and

$$V_2 = 24\pi \mu b_j \sum_{i \neq j} s_i \left(\frac{b_i}{r_{ij}} \right)^2 \mathbf{n}_{ij} + 12\pi \mu b_j \sum_{i \neq j} \left(\frac{b_i}{r_{ij}} \right)^3 B_{ij} \mathbf{v}_i. \tag{20}$$

In calculation of the third term we use similar identities and obtain

$$\begin{aligned}
V_3 &= -4\mu \sum_{i \neq j} \int_{\Gamma_i} \frac{\partial \psi_j}{\partial \mathbf{x}} \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \psi_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma \\
&= -4\mu \sum_{i \neq j} \int_{\Gamma_i} \left(-\frac{3}{4} \left(\frac{b_j}{r_{ij}} \right)^3 B_{ij} (I - \mathbf{n} \otimes \mathbf{n}) \right) \left(-\frac{2s_i}{b_i} \mathbf{n} + \frac{3}{2b_i} (I - 3\mathbf{n} \otimes \mathbf{n}) \mathbf{v}_i \right) d\Gamma \\
&= 12\pi \mu b_j^3 \sum_{i \neq j} \frac{b_i}{r_{ij}^3} B_{ij} \mathbf{v}_i.
\end{aligned}$$

Finally,

$$\begin{aligned}
\left(\frac{\partial \Phi}{\partial \mathbf{v}_j} \right)^T &\approx V_1 + V_2 + V_3 = 24\pi \mu b_j \mathbf{v}_j + 24\pi \mu b_j \sum_{i \neq j} s_i \left(\frac{b_i}{r_{ij}} \right)^2 \mathbf{n}_{ij} \\
&\quad + 12\pi \mu b_j \sum_{i \neq j} \left(\frac{b_i}{r_{ij}} \right)^3 B_{ij} \mathbf{v}_i + 12\pi \mu b_j^3 \sum_{i \neq j} \frac{b_i}{r_{ij}^3} B_{ij} \mathbf{v}_i.
\end{aligned} \tag{21}$$

We proceed now to find the viscous force in the equations governing the bubble oscillations:

$$\begin{aligned} \frac{\partial \Phi}{\partial s_j} = & -4\mu \sum_{i=1}^N \int_{\Gamma_i} \nabla \varphi_j \cdot \frac{\partial}{\partial r} (\nabla \varphi) d\Gamma \approx -4\mu \int_{\Gamma_j} \nabla \varphi_j \cdot \frac{\partial}{\partial r} (\nabla \varphi) d\Gamma \\ & - 4\mu \sum_{i \neq j} \int_{\Gamma_i} \nabla \varphi_j \cdot \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma. \end{aligned}$$

First, let us calculate the principal term

$$\begin{aligned} S_1 = & -4\mu \int_{\Gamma_j} \nabla \varphi_j \cdot \left(s_j \frac{\partial \nabla \varphi_j}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_j}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_j \right) d\Gamma \\ = & -4\mu \int_{\Gamma_j} \mathbf{n} \cdot \left(-\frac{2s_j}{b_j} \mathbf{n} + \frac{3}{2b_j} (I - 3\mathbf{n} \otimes \mathbf{n}) \mathbf{v}_j \right) d\Gamma = 32\pi \mu b_j s_j. \end{aligned} \quad (22)$$

The orthogonality of the Legendre polynomials and the first identity of (18) imply that for $i \neq j$ up to β^3

$$\int_{\Gamma_j} \nabla \varphi_j \cdot \frac{\partial \nabla \varphi_i}{\partial r} d\Gamma \approx 0, \quad \int_{\Gamma_j} \nabla \varphi_j \cdot \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i d\Gamma \approx 0.$$

Hence

$$S_2 = -4\mu \int_{\Gamma_j} \nabla \varphi_j \cdot \sum_{i \neq j} \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma \approx 0.$$

Now we have to calculate the last term

$$\begin{aligned} S_3 = & -4\mu \sum_{i \neq j} \int_{\Gamma_i} \nabla \varphi_j \cdot \left(s_i \frac{\partial \nabla \varphi_i}{\partial r} + \left(\frac{\partial}{\partial r} \left(\frac{\partial \boldsymbol{\psi}_i}{\partial \mathbf{x}} \right) \right)^T \mathbf{v}_i \right) d\Gamma \\ = & -4\mu \sum_{i \neq j} \int_{\Gamma_i} \left(-\frac{b_j^2}{b_i r_{ij}} \sum_{n=1}^2 \frac{2n+1}{n+1} \left(\frac{b_i}{r_{ij}} \right)^n P'_n(\cos \theta_{ji}) (I - \mathbf{n} \otimes \mathbf{n}) \mathbf{n}_{ji} \right) \left(\frac{3}{2b_i} (I - 3\mathbf{n} \otimes \mathbf{n}) \mathbf{v}_i - \frac{2s_i}{b_i} \mathbf{n} \right) d\Gamma. \end{aligned}$$

Since $(I - \mathbf{n} \otimes \mathbf{n}) \mathbf{n}_{ji} \cdot \mathbf{n} = 0$, and the polynomial $P'_2(x)$ is odd,

$$\begin{aligned} S_3 = & 9\mu b_j^2 \sum_{i \neq j} \frac{1}{b_i r_{ij}^2} \int_{\Gamma_i} (I - \mathbf{n} \otimes \mathbf{n}) \mathbf{n}_{ji} \cdot (I - 3\mathbf{n} \otimes \mathbf{n}) \mathbf{v}_i d\Gamma = 9\mu b_j^2 \sum_{i \neq j} \frac{1}{b_i r_{ij}^2} \int_{\Gamma_i} \mathbf{n}_{ji}^T (I - \mathbf{n} \otimes \mathbf{n}) \mathbf{v}_i d\Gamma \\ = & 24\pi \mu b_j^2 \sum_{i \neq j} \frac{b_i}{r_{ij}^2} \mathbf{n}_{ji} \cdot \mathbf{v}_i. \end{aligned}$$

Hence

$$\frac{\partial \Phi}{\partial s_j} \approx S_1 + S_2 + S_3 = 32\pi \mu b_j s_j + 24\pi \mu b_j^2 \sum_{i \neq j} \frac{b_i}{r_{ij}^2} \mathbf{n}_{ji} \cdot \mathbf{v}_i. \quad (23)$$

Finally, it follows from (21) and (23) that the drag forces are given by

$$\begin{aligned} \mathbf{F}_j^v = & -\frac{1}{2} \left(\frac{\partial \Phi}{\partial \mathbf{v}_j} \right)^T \approx - \left(12\pi \mu b_j \mathbf{v}_j + 12\pi \mu b_j \sum_{i \neq j} s_i \left(\frac{b_i}{r_{ij}} \right)^2 \mathbf{n}_{ij} \right. \\ & \left. + 6\pi \mu b_j \sum_{i \neq j} \left(\frac{b_i}{r_{ij}} \right)^3 B_{ij} \mathbf{v}_i + 6\pi \mu b_j^3 \sum_{i \neq j} \frac{b_i}{r_{ij}^3} B_{ij} \mathbf{v}_i \right) \end{aligned} \quad (24)$$

and

$$\mathbf{F}_j^s = -\frac{1}{2} \frac{\partial \Phi}{\partial s_j} \approx - \left(16\pi \mu b_j s_j + 12\pi \mu b_j^2 \sum_{i \neq j} \frac{b_i}{r_{ij}^2} \mathbf{n}_{ji} \cdot \mathbf{v}_i \right). \quad (25)$$

It is worth to note that expression (25) does not contain $O(\beta^3)$ terms: they vanish identically.

Formulae (24), (25) allow us to find the expression for the dissipation function Φ . Let us designate by \mathbf{U}_i^f the ambient velocity of fluid induced by the motion of other bubbles at the center of i th bubble:

$$\mathbf{U}_i^f = \sum_{k \neq i} \nabla_{\mathbf{x}_i} \left(-\frac{b_k^2 s_k}{r_{ik}} + \frac{b_k^3}{2} \nabla_{\mathbf{x}_i} \left(\frac{1}{r_{ik}} \right) \cdot \mathbf{v}_k \right). \quad (26)$$

Then the dissipation function Φ is given by

$$\Phi = \sum_{i=1}^N \mu \pi b_i (16s_i^2 + 12|\mathbf{v}_i - \mathbf{U}_i^f|^2).$$

This expression is exactly the same as that given by (4) for a single bubble, but in a fluid having the velocity \mathbf{U}_i^f at infinity. Direct calculations show that up to the terms of order β^3 it gives us the drag forces (24), (25). A similar form of the expression of the dissipation function was obtained by Wang and Smereka [20].

3.2. Equations of motion for a system of N bubbles

The buoyancy force and the surface tension can also be added through a simple changing of the Lagrangian of the system. It is sufficient to take L defined by (11), (12) in the form

$$L = T - U - \sum_{i=1}^N 4\pi \sigma b_i^2 - \sum_{i=1}^N \frac{4}{3} \pi \rho b_i^3 \mathbf{g} \cdot \mathbf{x}_i, \quad (27)$$

where σ is the surface tension coefficient and \mathbf{g} is the gravity. Finally, the equations of motion with Lagrangian (27) are:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \mathbf{v}_j} \right) - \frac{\partial T}{\partial \mathbf{x}_j} = & - \left(12\pi \mu b_j \mathbf{v}_j + 12\pi \mu b_j \sum_{i \neq j} s_i \left(\frac{b_i}{r_{ij}} \right)^2 \mathbf{n}_{ij} \right. \\ & \left. + 6\pi \mu b_j \sum_{i \neq j} \left(\frac{b_i}{r_{ij}} \right)^3 B_{ij} \mathbf{v}_i + 6\pi \mu b_j^3 \sum_{i \neq j} \frac{b_i}{r_{ij}^3} B_{ij} \mathbf{v}_i \right) - \frac{4}{3} \pi \rho b_j^3 \mathbf{g}, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial s_j} \right) - \frac{\partial T}{\partial b_j} = & 4\pi b_j^2 \left(p_g(b_j) - p_\infty - \frac{2\sigma}{b_j} - \rho \mathbf{g} \cdot \mathbf{x}_j - \frac{4\mu s_j}{b_j} - 3\mu \sum_{i \neq j} \frac{b_i}{r_{ij}^2} \mathbf{n}_{ji} \cdot \mathbf{v}_i \right). \end{aligned} \quad (28)$$

Here the kinetic energy T is given by (11). System (28) generalizes Eqs. (5) describing the motion of a single bubble.

4. Conclusion

In this paper we have presented a regular asymptotic procedure for the calculation of the drag forces on bubbles in a cloud of N compressible bubbles. By using the Levich approach, the viscous forces have been calculated explicitly up to order β^3 . The derivation has been based on the asymptotic expansion for the velocity potential with the same precision. Formulae obtained are in agreement with those found by Wang and Smereka [20]. A comparison with the lower order approximation obtained by Voinov and Golovin [8] shows that not all terms of order β^2 are presented in their formulae. Finally, a general system (28) describing the motion of N bubbles in the presence of real effects (viscous dissipation, gravity and surface tension) has been derived.

Acknowledgement

The research was accomplished when the authors were visiting the Mathematisches Forschungsinstitut Oberwolfach. V.T. was partially supported by the Russian Foundation of Basic Research, grant 04-01-00253, and by the Russian Ministry of Education, grant E02-4.0-21.

Appendix A. Approximate formulae for ψ_j and their derivatives

First, we begin with ψ_j . We have from (9), (10):

$$\begin{aligned}\psi_j|_{\Gamma_j} &\approx -\frac{1}{2}\left(\frac{b_j}{r_j}\right)^3(\mathbf{x}-\mathbf{x}_j)\Big|_{\Gamma_j} = -\frac{b_j}{2}\mathbf{n}, \\ \frac{\partial\psi_j}{\partial\mathbf{x}}\Big|_{\Gamma_j} &\approx -\frac{1}{2}\left(\frac{b_j}{r_j}\right)^3\left(I-3\frac{(\mathbf{x}-\mathbf{x}_j)\otimes(\mathbf{x}-\mathbf{x}_j)}{r_j^2}\right)\Big|_{\Gamma_j} = -\frac{1}{2}(I-3\mathbf{n}\otimes\mathbf{n}), \\ \frac{\partial}{\partial r}\left(\frac{\partial\psi_j}{\partial\mathbf{x}}\right)\Big|_{\Gamma_j} &\approx \frac{3}{2b_j}(I-3\mathbf{n}\otimes\mathbf{n}), \\ \psi_j|_{\Gamma_i} &\approx \frac{b_j^3}{2r_{ij}^2}\mathbf{n}_{ji} - \frac{1}{2}\frac{b_j^3}{r_{ij}^3}B_{ij}(\mathbf{x}-\mathbf{x}_i) - \frac{1}{4}\frac{b_j^3}{r_{ij}^3}\left(\frac{b_i}{r_i}\right)^3B_{ij}(\mathbf{x}-\mathbf{x}_i)\Big|_{\Gamma_i} = \frac{b_j^3}{2r_{ij}^2}\mathbf{n}_{ji} - \frac{3}{4}\frac{b_j^4}{r_{ij}^3}B_{ij}\mathbf{n}, \\ \frac{\partial\psi_j}{\partial\mathbf{x}}\Big|_{\Gamma_i} &\approx -\frac{1}{2}\frac{b_j^3}{r_{ij}^3}B_{ij} - \frac{1}{4}\frac{b_j^3}{r_{ij}^3}\frac{b_i^3}{r_i^3}B_{ij} + \frac{3}{4}\frac{b_j^3}{r_{ij}^3}\frac{b_i^3}{r_i^3}B_{ij}\frac{(\mathbf{x}-\mathbf{x}_i)\otimes(\mathbf{x}-\mathbf{x}_i)}{r_i^2}\Big|_{\Gamma_i} = -\frac{3}{4}\left(\frac{b_j}{r_{ij}}\right)^3B_{ij}(I-\mathbf{n}\otimes\mathbf{n}), \\ \frac{\partial}{\partial r}\left(\frac{\partial\psi_j}{\partial\mathbf{x}}\right)\Big|_{\Gamma_i} &\approx \frac{3}{4b_i}\left(\frac{b_j}{r_{ij}}\right)^3B_{ij}(I-3\mathbf{n}\otimes\mathbf{n}).\end{aligned}$$

Appendix B. Approximate formulae for φ_j and their derivatives

The formulae for φ_j following from (7)–(8) are:

$$\begin{aligned}\varphi_j|_{\Gamma_j} &\approx -\frac{b_j^2}{r_j}\Big|_{\Gamma_j} = -b_j, \\ \nabla\varphi_j|_{\Gamma_j} &\approx \frac{b_j^2}{r_j^3}(\mathbf{x}-\mathbf{x}_j)\Big|_{\Gamma_j} = \mathbf{n}, \\ \frac{\partial}{\partial r}(\nabla\varphi_j)\Big|_{\Gamma_j} &\approx -\frac{2}{b_j}\mathbf{n}, \\ \varphi_j|_{\Gamma_i} &\approx -\frac{b_j^2}{r_{ij}}\sum_{n=0}^2\left(\frac{r_i}{b_i}\right)^n\left(\frac{b_i}{r_{ij}}\right)^nP_n(\cos\theta_{ji}) - \frac{b_j^2}{r_{ij}}\sum_{n=1}^2\frac{n}{n+1}\left(\frac{b_i}{r_i}\right)^{n+1}\left(\frac{b_i}{r_{ij}}\right)^nP_n(\cos\theta_{ji})\Big|_{\Gamma_i} \\ &= -\frac{b_j^2}{r_{ij}}\sum_{n=0}^2\frac{2n+1}{n+1}\left(\frac{b_i}{r_{ij}}\right)^nP_n(\cos\theta_{ji}).\end{aligned}$$

Since

$$\begin{aligned}\cos\theta_{ji} &= \frac{(\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{n}_{ji}}{r_i}, \\ \nabla\cos\theta_{ji}|_{\Gamma_i} &= \frac{1}{r_i}\left(I-\frac{(\mathbf{x}-\mathbf{x}_i)\otimes(\mathbf{x}-\mathbf{x}_i)}{r_i^2}\right)\mathbf{n}_{ji}\Big|_{\Gamma_i} = \frac{1}{b_i}(I-\mathbf{n}\otimes\mathbf{n})\mathbf{n}_{ji},\end{aligned}$$

we have

$$\begin{aligned}\nabla\varphi_j|_{\Gamma_i} &\approx -\nabla\left(\frac{b_j^2}{r_{ij}}\sum_{n=1}^2\left(\frac{b_i}{r_{ij}}\right)^n\left(\left(\frac{r_i}{b_i}\right)^n+\frac{n}{n+1}\left(\frac{b_i}{r_i}\right)^{n+1}\right)P_n(\cos\theta_{ji})\right)\Big|_{\Gamma_i} \\ &= -\frac{b_j^2}{r_{ij}}\sum_{n=1}^2\left(\frac{b_i}{r_{ij}}\right)^n n\left(\frac{r_i^{n-1}}{b_i^n}-\frac{b_i^{n+1}}{r_i^{n+2}}\right)P_n(\cos\theta_{ji})\frac{(\mathbf{x}-\mathbf{x}_i)}{r_i}\end{aligned}$$

$$\begin{aligned}
& -\frac{b_j^2}{r_{ij}} \sum_{n=1}^2 \left(\frac{b_i}{r_{ij}} \right)^n \left(\left(\frac{r_i}{b_i} \right)^n + \frac{n}{n+1} \left(\frac{b_i}{r_i} \right)^{n+1} \right) P'_n(\cos \theta_{ji}) \\
& \frac{1}{r_i} \left(I - \frac{(\mathbf{x} - \mathbf{x}_i) \otimes (\mathbf{x} - \mathbf{x}_i)}{r_i^2} \right) \mathbf{n}_{ji} \Big|_{\Gamma_i} = -\frac{b_j^2}{b_i r_{ij}} \sum_{n=1}^2 \frac{2n+1}{n+1} \left(\frac{b_i}{r_{ij}} \right)^n P'_n(\cos \theta_{ji}) (I - \mathbf{n} \otimes \mathbf{n}) \mathbf{n}_{ji}.
\end{aligned}$$

Here “'” means the derivative of P_n with respect to $\cos \theta_{ji}$. Finally,

$$\frac{\partial}{\partial r} (\nabla \varphi_j) \Big|_{\Gamma_i} = -\frac{b_j^2}{b_i^2 r_{ij}} \sum_{n=1}^2 n(2n+1) \left(\frac{b_i}{r_{ij}} \right)^n P_n(\cos \theta_{ji}) \mathbf{n} + \frac{b_j^2}{b_i^2 r_{ij}} \sum_{n=1}^2 \frac{2n+1}{n+1} \left(\frac{b_i}{r_{ij}} \right)^n P'_n(\cos \theta_{ji}) (I - \mathbf{n} \otimes \mathbf{n}) \mathbf{n}_{ji}.$$

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